Inequality

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Let a,b,c,d and r be positive real numbers such that

r = (abcd)^¼ ≥ 1, prove that

1/(1 + a)² + 1/(1 + b)² + 1/(1 + c)² + 1/(1 + d)² ≥ 4/(1 + r)².

Remark. It is the **Problem 2969**, **Proposed by Vasile Cîrtoaje**, **University of Ploiesti**,**Romania**

Let $a,b,c,d,$ and r be positive real numbers such that $r = \sqrt[4]{abcd} \geq 1.$

Prove that

(**1**) $\dfrac{1}{(1+a)^2} + \dfrac{1}{(1+b)^2} + \dfrac{1}{(1+c)^2} + \dfrac{1}{(1+d)^2} \geq \dfrac{4}{(1+r)^2}.$

Solution by Arkady Alt , **San Jose** ,**California**, **USA.**

I suggest, as generalization of inequality (**1**), the following

Theorem.

For any natural $n \geq 2$ *positive* $a_1, a_2, \ldots, a_n$ *,such that* $a_1 a_2 \ldots a_n = r^n$ *inequality*

(**2**) $\dfrac{1}{(1+a_1)^2} + \dfrac{1}{(1+a_2)^2} + \ldots + \dfrac{1}{(1+a_n)^2} \geq \dfrac{n}{(1+r)^2}$

holds iff $r \geq \max\left\{\dfrac{1}{2}, \sqrt{n} - 1\right\}.$

Proof.

1.Necessety.

From supposition that this inequality is valid for all $a_1, a_2, \ldots, a_n > 0$ with

$a_1 a_2 \ldots a_n = r^n$ and by setting $a_1 = a_2 = \ldots = a_{n-1} = m,\ a_n = \dfrac{r^n}{m}, m \in \mathbb{N}$

we obtain inequality

$\dfrac{n-1}{(1+m)^2} + \dfrac{m^{2(n-1)}}{(m^{n-1}+r^n)^2} \geq \dfrac{n}{(1+r)^2}$ which holds for all natural m.

That yield $\lim\limits_{m\to\infty}\left(\dfrac{n-1}{(1+m)^2} + \dfrac{m^{2(n-1)}}{(m^{n-1}+r^n)^2}\right) = 1 \geq \dfrac{n}{(1+r)^2} \Rightarrow r \geq \sqrt{n} - 1.$

Since $\sqrt{n} - 1 > \dfrac{1}{2}$ for any natural $n > 2$ and $\sqrt{2} - 1 < \dfrac{1}{2}$ then case $n = 2$ should be considered separately. Another reason to do this is that case $n = 2$ we need as base of Math Induction in the proof of Sufficiency.

Suppose that for any $a, b > 0$ such that $ab = r^2$ inequality

(**3**) $\dfrac{1}{(1+a)^2} + \dfrac{1}{(1+b)^2} \geq \dfrac{2}{(1+r)^2}$

holds. Then $r \geq \sqrt{2} - 1$ accordingly to considered above general case.

Let $x := a + b.$ Then $x \geq 2r$ (inequality which provide the equivalence of the transition to (x,r) – notation) and inequality (**3**) becomes:

$\dfrac{2+2x+x^2-2r^2}{(1+r^2+x)^2} \geq \dfrac{2}{(1+r)^2} \Leftrightarrow (1+r)^2(2+2x+x^2-2r^2) - 2(1+x+r^2)^2 \geq 0 \Leftrightarrow$

(**4**) $(x-2r)(x(r^2+2r-1) + 2(r^3+r^2+r-1)) \geq 0.$

Since $r^2 + 2r - 1 \geq 0$ (that follows from $r \geq \sqrt{2} - 1$) and $x \geq 2r$ then inequality (**4**) must be

fulfilled for $x = 2r$. That is we obtain $2r(r^2 + 2r - 1) + 2(r^3 + r^2 + r - 1) \geq 0 \Leftrightarrow$
$2(2r - 1)(r + 1)^2 \Leftrightarrow r \geq 1/2$.
Also we can see that if $r \geq 1/2$ then $(x - 2r)(x(r^2 + 2r - 1) + 2(r^3 + r^2 + r - 1)) \geq$
$(x - 2r)(2r(r^2 + 2r - 1) + 2(r^3 + r^2 + r - 1)) = 2(x - 2r)(2r - 1)(r + 1)^2 \geq 0$.
Thus, (**3**) holds for for any $a, b > 0$ such that $ab = r^2$ iff $r \geq 1/2$.

2. **Sufficiency**.(Math. Induction by $n \geq 2$).
Since base of math. induction already proved we will pass to the step of math induction.
Let $a_1, a_2, \ldots, a_n, a_{n+1} > 0$ and $a_1 a_2 \ldots a_{n+1} = r^{n+1}$, where $r \geq \sqrt{n+1} - 1$.
Due to the symmetry of the inequality we can suppose that

$a_1 \geq a_2 \geq \ldots \geq a_n \geq a_{n+1} > 0$.

Let $x := \sqrt[n]{a_1 a_2 \ldots a_n}$ then $a_{n+1} = \dfrac{r^{n+1}}{x^n}$ and $x \geq a_{n+1} \Leftrightarrow x^{n+1} \geq r^{n+1} \Leftrightarrow x \geq r$.
Then $x \geq \sqrt{n+1} - 1 > \sqrt{n} - 1$ and by supposition of M.I. we have inequality
$\dfrac{1}{(1+a_1)^2} + \dfrac{1}{(1+a_2)^2} + \ldots + \dfrac{1}{(1+a_n)^2} \geq \dfrac{n}{(1+x)^2}$. Therefore,

$$\frac{1}{(1+a_1)^2} + \frac{1}{(1+a_2)^2} + \ldots + \frac{1}{(1+a_n)^2} + \frac{1}{(1+a_{n+1})^2} \geq \frac{n}{(1+x)^2} + \frac{x^{2n}}{(x^n + r^{n+1})^2},$$

and and it remains to prove that

$$\frac{n}{(1+x)^2} + \frac{x^{2n}}{(x^n + r^{n+1})^2} \geq \frac{n+1}{(1+r)^2} \text{ for all } x \geq r \geq \sqrt{n+1} - 1.$$

Let $h(x) := \dfrac{n}{(1+x)^2} + \dfrac{x^{2n}}{(x^n + r^{n+1})^2}$. Then

$$h'(x) = \frac{2n(x^{n+1} - r^{n+1})(x^{n+1}r^{n+1} + 3x^n r^{n+1} + r^{2n+2} - x^{2n-1})}{(1+x)^3 (x^n + r^{n+1})^3}.$$

Now everything depend on the behavior of polynomial

$P_n(x) := x^{n+1}r^{n+1} + 3x^n r^{n+1} + r^{2n+2} - x^{2n-1}$.

Note, that $x^{n+1}r^{n+1} + 3x^n r^{n+1} + r^{2n+2} - x^{2n-1} = 0 \Leftrightarrow r^{n+1} + \dfrac{3r^{n+1}}{x} + \dfrac{r^{2n+2}}{x^{n+1}} - x^{n-2} = 0$.
Let $\varphi(x) := r^{n+1} + \dfrac{3r^{n+1}}{x} + \dfrac{r^{2n+2}}{x^{n+1}} - x^{n-2}$.
Since $r \geq \sqrt{n+1} - 1 > \dfrac{1}{2}$ for $n \geq 2$, we have $P_n(r) = 2r^{2n+2} + 3r^{2n+1} - r^{2n-1} =$
$r^{2n-1}(2r^3 + 3r^2 - 1) = 2(r+1)^2(2r - 1) > 0 \Leftrightarrow \varphi(r) > 0$.
Since $\varphi(x)$ is a continuous function on $(0, \infty)$, $\varphi(\infty)\varphi(r) < 0$ and $\varphi(x)$ strictly decreasing on $[r, \infty)$, there is only one point $x_0 \in (r, \infty)$ such that
$\varphi(x_0) = 0 \Leftrightarrow P_n(x_0) = 0$.
Moreover $\varphi(x) > \varphi(x_0) = 0 \Leftrightarrow P_n(x) > 0$ for all $x \in [r, x_0)$ and
$0 = \varphi(x_0) > \varphi(x) \Leftrightarrow P_n(x) < 0$ for all $x \in (x_0, \infty)$.
Since $\min\limits_{x \in [r, x_0]} h(x) = h(r) = \dfrac{n}{(1+r)^2} + \dfrac{r^{2n}}{(r^n + r^{n+1})^2} = \dfrac{n+1}{(1+r)^2}$ and for
any $x \in [x_0, \infty)$ $h(x) > \lim\limits_{x \to \infty} h(x) = 1 \geq \dfrac{n+1}{(1+r)^2} = h(r)$ we obtain

$$\min_{x \in [r, \infty)} h(x) = h(r) = \frac{n+1}{(1+r)^2}.$$